A Dynamic Model of Order Execution and the Intraday Cost of Limit Orders

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Abstract

We developed a dynamic model of limit order in an order-driven market, wherein traders differ in their share valuations. Taking into consideration the traders’ learning process, and allowing variations in the conditional probability of limit order execution, we can analyze the dynamics of such order execution. Our results, which have interesting empirical implications, are closely related to existing literature on order sequences and order execution, and yield further insight into the dynamic process of order execution. Furthermore, the paper complements literature on transaction costs of limit orders, as this study shows that the intraday pattern of the cost of limit order submitted by uninformed traders is U-shaped.

Keywords: order-driven market, probability of execution, order sequences, intraday patterns, execution cost

I. Introduction

The successful development of electronic limit order trading platforms in almost all major stock markets worldwide has drawn increasing attention in the field of academic research on the order-driven market. Among the growing body of theoretical literature on order-driven markets, we find models that describe price formation process of limit orders (e.g., Glosten (1994), Seppi (1997), Parlour (1998), Foucault (1999)), or models that explore probability of limit order execution (e.g., Angel (1994)), and trader’s choice models of limit versus market orders (e.g., Handa and Schwartz (1996)). In this paper, we address the following questions: (1) What is the dynamic behavior of order execution by which uninformed traders can learn from? (2) What is the intraday pattern of the execution cost of limit orders?

Traders in order-driven markets face the dilemma of choosing which type of order to submit. A market order is executed with certainty at the quoted price. With
limit orders, a trader may improve price of execution, but could also run the risk of non-execution and face adverse selection risk if the order is executed. The cost of non-execution and adverse selection cost of execution is discussed extensively in the literature; however, little has been said on intraday behavior of costs. By analyzing the variations in the conditional probabilities of limit order execution, we could explore intraday behavior of non-execution cost and adverse selection risk of limit orders, thereby explaining the intraday pattern of liquidity and price volatility in an order-driven market.

Glosten (1994) derived equilibrium price schedule from an open limit order book. Using limit orders, he showed that while traders profit from liquidity-driven traders, they lose from information-driven price changes. Handa and Schwartz (1996) analyzed the rationale for limit order trading. They posited that traders who have minimal non-execution costs have an incentive to submit limit orders, while those who have high non-execution costs prefer to submit market orders, although this does not explicitly replicate traders’ decisions. Earlier theories on limit orders trading, such as the abovementioned studies, are mostly static models.

More recently, Parlour (1998) and Foucault (1999) developed dynamic models of limit order trading. In the study of Parlour (1998), endogenous probability of limit order execution depends on the state of the book, as well as an agent’s belief on upcoming order arrivals. Foucault (1999) provided a game theoretic model of price formation and order placement decisions. Foucault, Kadan, and Kandel (2001) developed a dynamic model of an order-driven market populated by discretionary liquidity traders. They found that the equilibrium pattern is determined by the “degree of impatience” of patient traders, their proportion in the population, and tick size.

However, while information asymmetry plays an important role in the real world, none of these dynamic models analyze on private information. Hence, their predictions may not always be consistent with empirical findings. For example, Parlour (1998) predicted that the conditional probability of a limit buy order, followed by the same type order, should be less than the conditional probability of a limit sell order when a limit buy order follows suit. In contrast, Ranaldo (2004) found the opposite empirical evidence with the Swiss Stock Exchange, pointing out that this may be due to information asymmetry being ruled out from Parlour’s model.

Handa, Schwartz, and Tiwari (2003) highlighted the issue of information asymmetry by extending the ideas of Foucault (1999) in a more general model wherein traders not only differ in share valuation, but also in information availability. They showed that the size of spread in an order-driven market is a function of adverse selection and the difference in valuation among investors. However, their model focused mainly on the determinants of bid-ask spread, and did not deal with the issue of order sequences and the dynamic behavior of order execution. Although Foucault (1999) and Handa, Schwartz, and Tiwari (2003)
improved on previous models with respect to information asymmetry, in their models, the probability of order arrival or order execution is non-dynamic; they contradict what many empirical papers observed.

Empirical evidence tells us that the probability of order arrival is not random. For example, orders are often followed by similar orders, which are referred to as the diagonal effect in Biais, Hillion, and Spatt’s (1995) study on limit orders in the Paris Bourse. Later, Al-Suhaibani and Kryznowski (2001) also found similar results in Saudi Stock Market. However, there is much less theoretical analysis on this issue. What is the dynamic behavior of order execution? Is there any intraday pattern in the execution cost of limit orders? Surprisingly, these questions have not been adequately addressed. Thus, the objective of this study is to develop a dynamic order execution model.

Our model is an extension of Foucault (1999) and Handa, Schwartz, and Tiwari (2003). It takes into consideration the information asymmetry of Handa, et al. This study extends the dynamics of conditional probability of order execution, and analyzes the dynamics of cost of limit orders in an order-driven market. In short, this paper differs from previous studies in two aspects: First, unlike previous studies wherein the probability of limit order execution is invariant to time, the conditional probability of limit order execution in our model is free to vary. Second, we explore the intraday patterns of transaction cost of limit orders. Since price formation processes and intraday patterns of spread and trading activities in specialist and dealer markets have already been extensively studied, this paper focuses on complementing previous literature on intraday transaction costs in an order-driven market.

As in the study of Handa, Schwartz, and Tiwari (2003), traders differ in their share valuation and advent of information. In static equilibrium, the determinants of price and spread are obtained. Static equilibrium results conform to other studies in terms of the determinants of bid ask spreads. Our results imply that the bid ask spread in the order-driven market increases together with the increase of the volatility of the asset, but decreases when the number of uninformed traders increases. The result is similar to the positive relation found between bid ask spread and adverse selection in many studies on quote-driven markets. In addition, results suggest that the relation between the difference in share valuation and bid ask spread is positive when there is no serious order imbalance. However, should there be an order imbalance, the greater the difference in share valuation among agents results to a smaller bid ask spread.

Results of the dynamic analysis show that, firstly, the probability of order execution is influenced by the structure of traders, expected value of the risky asset, and expected aggressiveness of other traders. This supports the empirical findings of Hollifield, Miller, and Sandas (2001), and Hollifield, Miller, Sandas, and Slive (2002). Second, contrary to Parlour (1998), we showed that the conditional probability of a buy (sell) order execution increases following the execution of a buy (sell) order. Finally, our result suggests that the intraday pattern of cost of
limit order submitted by uninformed traders is U-shaped.

The findings and implications of the model are closely related to other empirical and theoretical studies. For example, implications on order sequences help explain the empirical findings of Hamao and Hasbrouck (1995), Biais, Hillion, and Spatt (1995) and Ranaldo (2004). Moreover, the implications on bid-ask spread and adverse selection are similar to the theories of bid-ask spread in quote-driven markets, while determinants of the probabilities of execution are closely related to the findings of Foucault (1999), Foucault, Kadan, and Kandel (2001), and Handa, Schwartz, and Tiwari (2003).

This paper extends the literature of Foucault (1999) and Handa, Schwartz and Tiwari (2003) by supposing the presence of information asymmetry in the market and examining the dynamic adjustment of uninformed limit order traders on obtaining optimal limit. Under this model, as trading progresses, information is revealed and thus, the conditional probabilities of limit order execution expected by traders also vary, which in turn influences the limit on a limit order. It is hoped that with the improvement of these models, this paper can obtain more implications of a real market and further explain the dynamic order-driven market which cannot be observed by previous studies. The rest of the paper is organized as follows: Section 2 presents a model of a pure order-driven market; discusses static equilibrium and the determinants of price and bid-ask spread; and followed by the dynamic analysis. Section 3 discusses the implications of the model. Section 4 concludes the paper.

II. Theoretical Model

2.1 Model assumptions

Assumption 1: Asset Valuation

In every order-driven market, there is a single risk asset. The true value of the asset is a random variable \( \tilde{v} \), with \( N \) traders buying or selling this asset in the trading periods. Traders differ in their information about the asset value. Uninformed traders can observe public information of asset value; their trades are either driven by liquidity demand or influenced by noise. The other traders are the informed traders possessing private information on the asset value; their trades are information-driven. Assume uninformed traders believe that the value of the asset is uniformly distributed as \( \tilde{v} \sim U(V_L, V_H) \), and the expected asset value is

\[
E_u[\tilde{v}] = \frac{V_L + V_H}{2} = V_u.
\]

Informed traders believe that the value of the asset is
distributed as $\tilde{v} \sim U(V_l, V_h)$, and the expected asset value is $E_i[\tilde{v}] = \frac{V_l + V_h}{2} = V_i$. Due to the superiority of the private information, the precision of evaluation is higher for the informed trader, that is, $(V_h - V_l) < (V_H - V_L)$. The volatility of the asset value, which is based on public information

$$\frac{(V_H - V_L)^2}{12},$$

is greater than from private information

$$\frac{(V_h - V_l)^2}{12}.$$  

9 The true value of the asset becomes public by the end of the trading period.

**Assumption 2: Trading periods**

The time horizon is one normal trading day. A trading day is divided into discrete time intervals denoted by $t$, $t=1,2,3,...,\tilde{T}$. We assume that payoff time $\tilde{T}$ is a random value.\(^{10}\) At time $t$, the probability that the trader’s expected trading process would stop and the payoff of the asset is realized, $(1-\rho_t) > 0$, where $\rho_t$, is the probability of the continuation of the trading process. Furthermore, $(1-\rho_t)$ is an increasing function of $t$, that is, $\frac{\partial(1-\rho_t)}{\partial t} > 0$.

**Assumption 3: Order placement strategies**

Traders arrive sequentially to trade one share of the asset. Upon arrival, an uninformed trader can choose to place a limit order or a market order. A limit order is held until the next trader arrives, at which point it is either executed or expired.

**Assumption 4: Trading behavior**

The behavior of the two types of traders is described in more detail as follows:

a. Uninformed trader: Suppose there are $U$ uninformed traders out of the $N$ traders. In addition to public information, these traders may be influenced by noise, and as such, would have different reservation prices for the asset. Assume that there are $U_b$ uninformed traders who think the asset value is equal to
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\[ V_{U_s} = \int_{V_s}^{V_H} \tilde{V} f(\tilde{V}) d\tilde{V} = V_u + \varepsilon, \quad \varepsilon > 0; \] they are the buyers. Meanwhile, there are \( U_s \) uninformed traders who think the asset value is equal to \( \varepsilon \); they are the sellers. If noise \( \varepsilon \) only influences the range of the volatility of asset value and not the distribution of the value, then

\[ \varepsilon = \frac{1}{4} (V_H - V_L). \]

b. Informed traders: Suppose there are \( I \) informed traders out of \( N \) traders. They gain profits through superior information. When \( E_i[\tilde{V}] = V_i > E_u[\tilde{V}] = V_u \), they will choose to buy the asset; when \( V_i < V_u \), they will sell the asset. If \( V_i = V_u \), they will not enter the market. Since private information is short-lived when there is competition among informed traders, we assume that the informed trader only submits market orders.

Assume that traders are risk-neutral and expecting utility maximizers. For a specified price \( P \), the expected utility of a buy order is \( \lambda (E[V] - P) \) and the expected utility of a sell order is \( \lambda (P - E[V]) \), where \( \lambda \) is the probability of execution of the order and \( E[V] \) is the expected asset value.

2.2 Equilibrium of bid-ask prices and spread

Once a trader submits an order, the probability of limit order execution will depend on the information they owned. There are three possible relations between public and private information, specifically, \( V_i > V_u, \quad V_i = V_u, \) and \( V_i < V_u \), with the probability of each is 1/3. Since a limit order can only be executed when a market order arrives, the trader’s placement strategies are interrelated. Assume the expected probability of a buy limit order placed by other uninformed traders is \( \pi_1 \), and the expected probability of a sell limit order placed by other uninformed traders is \( \pi_2 \). Thus, \( \pi_1 \) and \( \pi_2 \) can describe the aggressiveness of the orders. For example, if there are more aggressive traders in the market, \( \pi_1 \) or \( \pi_2 \) would be small.
A more straightforward approach is as follows:

a. If \( V_i > V_u \), there are \( I+U_b \) buyers and \( U_s \) sellers. The probability of buyer arrival is \( P_1 = \frac{I}{I+U} \), and the probability of sellers entering the market is \( (1-P_1) \). The proportion of informed buyers to total buyers is \( k_1 = \frac{I}{I+U_b} \). In this case, there are no informed sellers.

b. If \( V_i = V_u \), there are \( U_b \) buyers and \( U_s \) sellers. The probability of buyer arrival is \( P_2 = \frac{U}{U} \), and the probability of sellers entering the market is \( (1-P_2) \). In this case, informed traders do not trade.

c. If \( V_i < V_u \), there are \( U_b \) buyers and \( I+U_s \) sellers. The probability of buyers entering the market is \( p_3 = \frac{U_b}{I+U} \), and the probability of sellers entering the market is \( (1-p_3) \). The proportion of informed sellers to total sellers is \( k_2 = \frac{I}{I+U_s} \). In this case, there are no informed buyers.

[Insert Figure 1]

This tree describes the possible paths faced by uninformed traders, where \( V_i \) indicates the expected asset value of informed traders, and \( V_u \) indicates the expected asset value of uninformed traders; \( p_1 \), \( p_2 \), and \( p_3 \) are the probabilities of buyers entering the market; \( k_1 \) and \( k_2 \) are the probabilities of informed trading; and \( \pi_1 \) and \( \pi_2 \) are the expected probability of the limit buy and limit sell orders placed by other uninformed traders.

Upon arrival, an uninformed trader can choose to submit a limit or a market order. Based on the order path in Figure 1, the expected utility of the uninformed trader is analyzed as follows:

First, consider an uninformed buyer who arrives at time \( t \) and places a limit
order. Let \( B_t \) be the bid price. The order will be executed if (1) the trading interval does not stop before the arrival of the next trader, (2) the next trader is a seller, and (3) the next trader submits a market order. The expected utility of this buyer is:

\[
E(U) = \frac{\rho}{3} \{ (1-\pi_2)[3 - (p_1 + p_2 + p_3) + p_3 k_2] \} \{ (V_u + \varepsilon) - B_t \} \tag{1}
\]

Let \( A_{t-1} \) be the ask price of an uninformed seller who arrives at time \( t-1 \). The expected utility of an uninformed buyer arriving at time \( t \) and submitting a market buy order would be

\[
E(U) = (V_u + \varepsilon) - A_{t-1} \tag{2}
\]

For a buyer wanting to become indifferent to a market order and a limit order, \( B_t \), they must satisfy the following equality:

\[
(V_u + \varepsilon) - A_{t-1} = \rho \frac{1}{3} \{ (1-\pi_2)[3 - (p_1 + p_2 + p_3) + p_3 k_2] \} \{ (V_u + \varepsilon) - B_t \} \tag{3}
\]

Similarly, let \( A_t \) be the ask price of a limit sell. The expected utility of this seller is

\[
E(U) = \frac{\rho}{3} \{ (1-\pi_1)[p_1(1-k_1) + p_2 + p_3] + p_1 k_1 \} \{ A_t - (V_u - \varepsilon) \} \tag{4}
\]

Let \( B_{t-1} \) be the bid price of a buyer arriving at time \( t-1 \). Hence, the expected utility of a seller arriving at time \( t \) and submitting a market sell order is

\[
E(U) = B_{t-1} - (V_u - \varepsilon) \tag{5}
\]

For a seller wanting to become indifferent to a market and a limit order, \( A_t \) should satisfy

\[
B_{t-1} - (V_u - \varepsilon) = \rho \frac{1}{3} \{ (1-\pi_1)[p_1(1-k_1) + p_2 + p_3] + p_1 k_1 \} \{ A_t - (V_u - \varepsilon) \} \tag{6}
\]

A stationary solution can be derived from Equations (5) and (6); it does not depend on time.\(^{11}\) The equilibrium is characterized by the optimal bid and ask prices \( \{ A^*, B^* \} \), as established in Proposition 1.
Proposition 1.

Given the values for $I, U_b, U_s, V_H, V_L, \pi_1, \text{ and } \pi_2$, the equilibrium bid and ask prices can be expressed as

$$B^* = V_u + \frac{2y \epsilon - (1 + xy) \epsilon}{(1 - xy)}$$

(7)

$$A^* = V_u + \frac{(1 + xy) \epsilon - 2x \epsilon}{(1 - xy)}$$

(8)

For the bid ask spread, it is

$$s = A^* - B^* = 2 \epsilon \left\{ \frac{(1-x)(1-y)}{(1-xy)} \right\} \geq 0$$

(9)

Where $p_1 = \frac{I + U_b}{I + U}$, $p_2 = \frac{U_b}{U}$, $p_3 = \frac{U_b}{I + U}$, $k_1 = \frac{I}{I + U_b}$, $k_2 = \frac{I}{I + U_s}$

$$x = \rho \frac{1}{3} \left\{ (1 - \pi_2) [3 - (p_1 + p_2 + p_3) + p_3 k_1] \right\}, \quad 0 \leq x \leq 1$$

(10)

$$y = \rho \frac{1}{3} \left\{ (1 - \pi_1) [p_1 (1 - k_1) + p_2 + p_3] + p_3 k_1 \right\}, \quad 0 \leq y \leq 1$$

(11)

(For proof, see Appendix A.)

Here, $x$ is the average probability of execution of a limit buy order, and $y$ is the average probability of execution of a limit sell order. In equilibrium, the relationships between the limit price and the unconditional expected value of asset are as follows:

When $2y - 1 - xy \geq 0$, then $A^* \geq B^* \geq V_u$. (12a)

When $1 + xy - 2x \geq 0$ and $2y - 1 - xy \leq 0$, then $A^* \geq V_u \geq B^*$. (12b)

When $1 + xy - 2x \leq 0$, then $V_u \geq A^* \geq B^*$. (12c)

Equations 12a, 12b, and 12c delineate how the level of limit prices is affected by the execution probabilities, $x$ and $y$. The three conditions of $x$ and $y$ are illustrated in Figure 2.
Figure 2 shows that when the values of x and y are in the area of \( \triangle ABC \) (y is far greater than x), then \( A^* \geq B^* \geq V_u^* \); when the values of x and y are in the area of \( \square BCDE \) (x is close to y), then \( A^* \geq V_u^* \geq B^* \); and when the values of x and y are in the area of \( \triangle DEF \) (x is far greater than y), then \( V_u^* \geq A^* \geq B^* \).

From Equations (10) and (11), we know that x is the average probability of execution of a limit buy order, and y is the average probability of execution of a limit sell order. The intuition of the relationship between the values of x and y, and the limit prices, is straightforward. When x is high, indicating larger numbers of sellers or more aggressive sellers, the bid and ask prices will be lower. When y is high, indicating larger number of buyers or more aggressive buyers, the bid and ask prices will be higher. However, despite being straightforward, the conditions shown in (12) could assist in the discussions of determinants for equilibrium bid-ask prices.

2.3 The determinants of the equilibrium bid-ask prices

We analyzed the determinants of equilibrium bid-ask prices in three cases.

Case 1. \( 1 + xy - 2x \geq 0, \; 2y - 1 - xy \leq 0 \) (\( A^* \geq V_u^* \geq B^* \))

Proposition 2.

When \( 1 + xy - 2x \geq 0 \) and \( 2y - 1 - xy \leq 0 \), it can be shown that

\[
\frac{\partial A^*}{\partial \varepsilon} \geq 0, \quad \frac{\partial A^*}{\partial x} \leq 0, \quad \frac{\partial A^*}{\partial y} \geq 0, \quad \frac{\partial B^*}{\partial \varepsilon} \leq 0, \quad \frac{\partial B^*}{\partial x} \leq 0, \text{ and } \frac{\partial B^*}{\partial y} \geq 0.
\]

(For proof, see Appendix B.)

The size of \( \varepsilon \) indicates noise in public information. When the volatility of asset value rises, \( \varepsilon \) rises, and the risk of adverse selection born by uninformed traders increases. The uninformed trader requires a larger premium in this case. Consequently, ask price increases while bid price decreases.

When the average probability of execution for a limit buy order is large, the non-execution risk of the buyers is lower than from the sellers. Consequently, the
bid price decreases as buyers require higher premium, while ask price decreases as sellers seek to lower their risk of execution.

When the average probability of execution for a limit sell order is large, the non-execution risk of buyers are higher than that from the sellers. Consequently, ask price increases as sellers require higher premium, while bid price increases as buyers seek to reduce their risk of execution.

Case 2. \(2y - 1 - xy \geq 0 \ (A^* \geq B^* \geq V_u)\)

Proposition 3.

If \(2y - 1 - xy \geq 0\), then \(\frac{\partial A^*}{\partial \varepsilon} \geq 0\), \(\frac{\partial A^*}{\partial x} \leq 0\), \(\frac{\partial A^*}{\partial y} \geq 0\) and \(\frac{\partial B^*}{\partial \varepsilon} \geq 0\),

\[\frac{\partial B^*}{\partial x} \leq 0, \quad \text{and} \quad \frac{\partial B^*}{\partial y} \geq 0.\]

(For proof, see Appendix C.)

In this case, the probability of execution of limit sell order \(y\) is far greater than \(x\), and the non-execution risk is very high for the uninformed traders with high valuations. As \(\varepsilon\) increases, the buyer’s valuation rises in order to reduce the risk of non-execution they need to increase bid price. Except for the relationship between \(\varepsilon\) and \(B^*\), all other relationships hold, as in Proposition 2.

Case 3. \(1 + xy - 2x \leq 0 \ (V_u \geq A^* \geq B^*)\)

Proposition 4.

If \(1 + xy - 2x \leq 0\), then \(\frac{\partial A^*}{\partial \varepsilon} \leq 0\), \(\frac{\partial A^*}{\partial x} \leq 0\), \(\frac{\partial A^*}{\partial y} \geq 0\) and \(\frac{\partial B^*}{\partial \varepsilon} \leq 0\),

\[\frac{\partial B^*}{\partial x} \leq 0, \quad \text{and} \quad \frac{\partial B^*}{\partial y} \geq 0.\]

(For proof, see Appendix D.)

In this case, \(x\), the probability of execution of limit buy order is far greater than \(y\), and the non-execution risk is very high for uninformed traders with low valuations. As such, when \(\varepsilon\) rises, the seller’s valuation of asset falls; to reduce
the risk of non-execution, they need to lower the ask price. Except for the relationship between $\varepsilon$ and $A^*$, all other relationships hold, as in Proposition 2.

The following comparative statics show the relationship between the bid-ask spread and its determinants.

**Proposition 5.**

The relationships between the bid-ask spread and $\varepsilon$, $x$, and $y$ are:

$$\frac{\partial s}{\partial \varepsilon} \geq 0,$$

$$\frac{\partial s}{\partial x} \leq 0,$$ and $$\frac{\partial s}{\partial y} \leq 0.$$

(For proof, see Appendix E.)

As the volatility of asset value rises, $\varepsilon$ rises, and the risk of adverse selection perceived by uninformed traders also increases. Traders require higher premium. Consequently, the ask price increases and the bid price decreases, and the bid-ask spread widens.

When the number of uninformed traders increases, both $x$ and $y$ increase, and the risk of adverse selection perceived by uninformed traders decreases. Traders require less premium; consequently, the ask price decreases and the bid price increases, causing the bid-ask spread to narrow.

**2.4 Dynamic analysis**

In the previous static equilibrium analysis, we find that the levels of $x$ and $y$ affect equilibrium price and bid-ask spread, which is consistent with other studies.\(^{14}\) In the real world, however, the probability of order execution is not constant. In this section, we shall then proceed to the dynamic analysis.

In the above analysis, we assumed there are three equally possible conditions between public and private information: $V_i > V_a$, $V_i = V_a$, and $V_i < V_a$. However, as trading progresses, the expectation of the uninformed traders will adjust according to order flow. For example, if only sell limit orders are executed, the uninformed traders cannot perceive an equal probability for three situations; the
equal probability assumption is elaborated in the following analysis.

If the last executed limit order is a sell order, the conditional probabilities of the three market situations are

\[
\delta_{11} | I_{t-1} = \text{Prob}(V_i > V_a | EL_{t-1} = SO) = \frac{\delta_{11} [p_1 k_1 + p_1 (1-k_1)(1-\pi_1)]}{y_{t-1}}
\]

(13)

\[
\delta_{21} | I_{t-1} = \text{Prob}(V_i = V_a | EL_{t-1} = SO) = \frac{\delta_{21} [p_2 (1-\pi_1)]}{y_{t-1}}
\]

(14)

\[
\delta_{31} | I_{t-1} = \text{Prob}(V_i < V_a | EL_{t-1} = SO) = \frac{\delta_{31} [p_1 (1-\pi_1)]}{y_{t-1}}
\]

(15)

If the last executed limit order is a buy order, then the conditional probabilities of the three market situations are

\[
\lambda_{11} | I_{t-1} = \text{Prob}(V_i > V_a | EL_{t-1} = BO) = \frac{\lambda_{11} [1-p_1 (1-\pi_2)]}{x_{t-1}}
\]

(16)

\[
\lambda_{22} | I_{t-1} = \text{Prob}(V_i = V_a | EL_{t-1} = BO) = \frac{\lambda_{22} [1-p_2 (1-\pi_2)]}{x_{t-1}}
\]

(17)

\[
\lambda_{32} | I_{t-1} = \text{Prob}(V_i < V_a | EL_{t-1} = BO)
\]

\[
= \frac{\lambda_{32} [k_2 (1-p_2) + (1-k_2) (1-p_2) (1-\pi_2)]}{x_{t-1}}
\]

(18)

where \( I_{t-1} \) denotes the information set at time t-1; Prob denotes probability; \( EL_{t-1} \) is the executed limit order at time t-1; SO is the sell order; and BO is the buy order.

Given that the last executed limit order is a sell order, the conditional probability of execution of a limit sell order is

\[
E(y_i | EL_{t-1} = SO) = \rho_1 \left\{ \delta_{11} y_{t-1} \times [p_1 k_1 + p_1 (1-k_1)(1-\pi_1)] + \delta_{21} y_{t-1} \times p_2 (1-\pi_1) + \delta_{31} y_{t-1} \times p_1 (1-\pi_1) \right\}
\]

(19)

Given that the last executed limit order is a buy order, the conditional probability of execution of a limit sell order is
Given that the last executed limit order is a sell order, the conditional probability of execution of a limit buy order is

$$E(\gamma_t | EL_{t-1} = BO) = p_1 \{ \lambda_{t,1} L_{t-1} \times \left[ (p_2 k_1 + p_1 (1-k_1)(1-\pi_2)) \right] + \lambda_{t,2} L_{t-1} \times p_2 (1-\pi_1) + \lambda_{t,3} L_{t-1} \times p_3 (1-\pi_1) \}$$

(20)

Given that the last executed limit order is a buy order, the conditional probability of execution of a limit buy order is

$$E(\gamma_t | EL_{t-1} = SO) = p_1 \{ \delta_{t,1} L_{t-1} \times \left[ (1-p_1)(1-\pi_2) \right] + \delta_{t,2} L_{t-1} \times \left[ (1-p_2)(1-\pi_2) \right]$$

$$+ \delta_{t,3} L_{t-1} \times \left[ k_2 (1-p_3) + (1-k_2)(1-p_3)(1-\pi_2) \right] \}$$

(21)

From Equations (13) to (22), we find that the average probability of execution is influenced by the probability of the stops of the trading process (1-\(\rho_1\)), the structure of traders (\(I,U_b,U_s\)), the expected value of asset (\(\delta_{t,1},\delta_{t,2}\)), and the aggressiveness of traders (\(\pi_1,\pi_2\)). Hence, the optimal limit price is also influenced by these factors.

2.5 Order sequences

Previous studies have found a conditional order flow pattern. Take this scenario: After the arrival of a limit buy order at the best bid price, the incoming order is most likely to be the same order type. This is called the diagonal effect. This result is first documented by Biais, Hillion, and Spatt (1995) in the Paris Bourse, followed by Al-Suhaibani and Kryznowski (2001) on the Saudi Stock Market. Biais, Hillion, and Spatt (1995) offered three explanations to the diagonal effect: strategic order splitting, trade imitation, or similar reaction to information event.

The findings on order sequences of this model are consistent with the documented diagonal effect. From Equations (19) to (22), we can see that the conditional probability of order execution, followed by the same type order, is higher than the conditional probability of order execution, which is then again
followed by a different type order. If the last executed order is a sell order, then the conditional expected probability of $V_i > V_u$ increases. As a result, the probability of execution of limit sell order increases. If the last executed order is a buy order, then the conditional expected probability of $V_i > V_u$ decreases. As a result, the probability of execution of limit buy order increases. The following equations define the “diagonal effect”:

\[ E( y_i | E_{t-1} = SO ) > E( y_i | E_{t-1} = BO ) \] (23)

\[ E( x_i | E_{t-1} = SO ) < E( x_i | E_{t-1} = BO ) \] (24)

### 2.6 The cost of limit order trading

The cost of limit order trading has two components: cost of non-execution risk and cost of adverse selection risk. Conditional on private information, the non-execution cost of a limit buy order of the uninformed trader is

\[ 1 - \left\{ \rho \times \theta_1 | I_{t-1} \times [(1 - p_1)(1 - \pi_2)] \right\} (V_i - B_i) \] (25)

The cost of adverse selection risk of a limit buy order is

\[ \rho_1 \left\{ \theta_3 | I_{t-1} \times [k_2 (1 - p_3) + (1 - k_2)(1 - p_3)(1 - \pi_2)] \right\} (V_i - B_i) \] (26)

Conditional on private information, the non-execution cost of a limit sell order of the uninformed trader is

\[ 1 - \left\{ \rho \times R \times \theta_3 | I_{t-1} \times p_3 (1 - \pi_1) \right\} (A_i - V_i) \] (27)

The cost of adverse selection risk of a limit sell order is

\[ \rho_1 \left\{ \theta_3 | I_{t-1} \times [p_1 k_1 + p_1 (1 - k_1)(1 - \pi_1)] \right\} (V_i - B_i) \] (28)

Here, $\theta$ are the conditional probabilities of the three market situations. If the last executed limit order is a buy order, then $\theta_{t-1} = \lambda_{t-1}$; if the last executed limit order is a sell order, then $\theta_{t-1} = \delta_{t-1}$, $t = 1, 3$.

From Equations (25) and (27), we find that the cost of non-execution risk is an increasing function of $t$ since $(1 - \rho_t)$ is an increasing function of $t$. Meanwhile, from Equations (26) and (28), we find that the cost of adverse selection risk is a decreasing function of $t$. Since information is disclosed as trading proceeds, the expectation and limit price of uninformed traders will adjust to the order flow
accordingly to reduce risk in adverse selection. The limit price of uninformed traders will then be more efficient, such that 

\[(V_i - B_t (\theta_{3t} \vert I_{t-1})) \leq (V_i - B_t (\theta_{3t} \vert I_{t-1})) \] and 
\[(A_t (\theta_{1t} \vert I_{t-1}) - V_i) \leq (A_t (\theta_{1t} \vert I_{t-1}) - V_i), \] where \(\tau < t.\)

Therefore, the intraday pattern of the cost of limit order submitted by uninformed traders is U-shaped.

**III. Model Implications and Relations to the Literature**

Implication 1: The equilibrium limit of buy and sell prices by liquidity traders are equal to the unconditional expected value of the asset.

If the uninformed traders are not different in terms of share valuation, such that the noise of the expected asset value \(\varepsilon\) is zero, then from Equations (7) and (8), the equilibrium limit of buy and sell prices are equal to \(V_u\). Hence, if trades are driven by liquidity demand, then the optimal limit price is equal to the unconditional expected value of the asset.

Implication 2: The bid-ask spread is an increasing function of the volatility of the unconditional expected value of the asset.

Implication 2 is consistent with theoretical models of the bid-ask spread in dealer markets (see, for example, Copeland and Galai, 1983; Easley and O’Hara, 1987) and in order-driven markets (see, for example, Handa et al., 2003). They all find that the bid-ask spread increases in information asymmetry and during uncertainties in the degree of asset value.

Implication 3: The bid-ask spread is a decreasing function of the number of uninformed traders.

Using Proposition 5, when the number of uninformed traders increases, both \(x\) and \(y\) increase, the risk of adverse selection perceived by uninformed traders decreases, and the traders will require less premium. Consequently, ask price decreases while bid price increases, causing the bid-ask spread to narrow.

Glosten and Milgrom (1985) suggested that the bid-ask spread in dealer markets contains an informational component, while the market-maker loses to informed traders on average but recoups these losses in noise trades. The market-maker must trade off the reduction in losses to the informed trader from a wider spread against opportunity cost in terms of gaining profits from trading with uniformed traders, as well as in reservation prices inside the spread. The situation faced by the uninformed limit order traders is similar to the market-maker; therefore, when the number of uninformed traders increase, the risk of adverse selection decreases, and the bid-ask spread of limit prices is narrowed.

Implication 4: When there is no order imbalance in the market, greater asset value noise leads to lower bid price and higher ask price. If there are far more
sellers than buyers, then increasing the asset noise will lead to a lower ask price, and if there are far more buyers than sellers, the greater noise will lead to a higher bid price.

Handa et al. (2003) showed that the spread in an order-driven market is highest when the buy and sell orders are balanced, but the spread is minimized when there is a large order imbalance. In their model, only the risk of adverse selection is considered. In our model, the cost of limit order includes adverse selection as well as non-execution risk. We showed that the structure of traders not only influences the size of the spread, but also influences the relation between noise and limit prices.

Implication 5: The probabilities of limit order execution are influenced by the probability of the trading process stops, the structure of traders, the expected value of asset, and the expected aggressiveness of the other traders.

From Equations (13) to (22), we find that the average probability of execution is influenced by the probability of the trading process stops \( (1 - \rho_1) \), the structure of traders \( (\text{sb}_U, U, \text{sb}_b) \), the expected value of asset \( (\delta_u, \lambda_u) \), and the aggressiveness of traders \( (\pi_1, \pi_2) \). Hence, the optimal limit price is also influenced by the above factors.

The determinants of the probabilities of execution in this paper are closely related to the findings of Foucault (1999), Foucault, Kadan, and Kandel (2001), and Handa, Schwartz, and Tiwari (2003). It further supports the empirical findings of Hollifield, Miller, and Sandas (2001), and Hollifield, Miller, Sandas, and Slive (2002).

Implication 6: The conditional probability of limit order execution, which is followed by the same type order, should be higher than the conditional probability of limit order execution, and then followed again by a different type order.

Previous studies have found a conditional order flow pattern: For instance, after the arrival of a limit buy order at the best bid price, the incoming order is most likely to be the same order type. This is called the diagonal effect (see Biais, Hillion, and Spatt, 1995). The findings on conditional “execution” order flow pattern of this model is consistent with previous studies. Furthermore, the implication supports the empirical findings of Ranaldo (2004), which is contrary to Parlour (1998).

Implication 7: The intraday pattern of the cost of limit order submitted by uninformed traders is U-shaped.

Admati and Pfleiderer (1988) developed a theory to explain the concentration of trading at the open and close of a day. They proposed that discretionary liquidity and informed trading will concentrate at the open and the close due to higher liquidity trading in these periods. Following these thought, suppose that discretionary liquidity traders choose trading time to minimize the trading cost, then the commonly observed U-shaped pattern of trading activities should suggest an inverse U-shaped pattern of trading cost, which is contrary to our findings. However, studies have also found that the variance of price and returns follow a
U-shaped pattern.\textsuperscript{16} Glosten (1994) argues that limit order traders gain profit from liquidity-driven traders, but they lose from information-driven price changes. Therefore, if the large price changes at the beginning and the end of the trading day are attributed to informed trading, then the trading cost of the uninformed limit order traders will be large at the beginning and the end of the trading day. This has been predicted by our model.

Foster and Viswanathan (1990) developed an adverse selection model and examined interday variations in volume, variance, and adverse selection costs. They found that trading costs and variance of price changes are highest on Mondays. Our findings on adverse selection costs are similar in that the adverse selection costs of uninformed limit order traders are large at the beginning of the trading periods. In addition to intraday patterns on volume and volatility, many empirical studies have documented that the bid-ask spread is widest at the opening of a day.\textsuperscript{17} Our model helps explain this phenomenon by exploring the changes in the trading costs of the uninformed traders in a trading day.

IV. Conclusion

We developed an information asymmetric model in which the conditional probability of order execution and the cost of limit orders are dynamic.

Several interesting implications of this model are observed that are closely related to existing empirical and theoretical studies. For example, the implication on order sequences is consistent with the empirical evidences in the studies of Hamao and Hasbrouck (1995), Biais, Hillion, and Spatt (1995), and Ranaldo (2004). The implication on bid-ask spread in order-driven markets is similar to the theories on bid-ask spread in dealer markets. Furthermore, our result complements the literature on trading costs of limit order. We showed that the intraday pattern of the cost of limit order that is submitted by uninformed traders is U-shaped. Our findings may shed more light on the dynamics of order execution, and the intraday pattern of market performances in order-driven markets.

V. Endnotes

\textsuperscript{1} Although there are many empirical studies focusing on the trading cost of limit order, none has discussed intraday pattern. Some focused on the costs and determinants of order aggressiveness (e.g., Keim and Madhavan (1997), and Giffiths, Smith, Turnbull, White (2000)); others on the comparison of trading costs for different stocks (e.g., Bessembinder and Kaufman (1997), Jones and Lipaon (1997)); and others on the survival analysis of limit order execution times and their determinants (e.g., Lo, MacKinlay, Zhang (2002)).

\textsuperscript{2} Handa, Schwartz, and Tiwari (2003) described the unconditional probabilities of the arrival of uninformed traders and the limit order execution, but not the
conditional dynamics of order execution.

The foregoing research examined the traders’ choice between limit and market orders; they did not discuss intraday pattern. Moreover, in the last few years, several empirical studies have been devoted to the issue of the determinants of trader’s order choice, as well as probabilities and times of limit order execution. For example, using data from Paris Bourse, Biais, Hillion, and Spatt (1995) found evidence that traders submit more market orders when the order book is relatively full, and more limit orders when the order book is relatively empty. Harris and Hasbrouck (1996) analyzed the profitability of alternative order placement strategies in different market conditions. Hollifield, Miller, and Sandas (2001) analyzed order placement strategies in a limit order market using data on the order flow from the Stockholm Stock Exchange. Lo, MacKinlay, and Zhang (2001) developed econometric models of limit order execution times using survival analysis, and estimated them with actual limit order data. Bae, Jang, and Park (2003) examined a trader’s order choice between market and limit orders using a sample of orders submitted through NYSE SuperDot.

For example, Copeland and Galai (1983), Glosten and Milgrom (1985), and Easley and O’Hara (1987).

Their findings implied that variation in the composition of the order flow can be explained by the variation in the relative profitability of alternative order choices and the movements in the common value of the asset.

Hollifield, Miller, Sandas, and Slive (2002) found that a trader’s optimal order submission changes with market conditions.

Hamao and Hasbrouck (1995) found order persistence and suggested that order continuation may depend on information motives. Biais et al. (1995) explained this in greater detail. They reported that the most likely incoming order type would be the same order type that has just arrived. This phenomenon may be a result of order splitting, trading imitation, and the same response to information. Ranaldo (2004) observed the sequence of a trade and found subsequent orders in the same direction in the Swiss Stock Exchange.

This informational source of the spread was first suggested by Bagehot (1971) and formally analyzed by Copeland and Galai (1983). Glosten and Milgrom (1985) used a formal model to show how the spread arises from adverse selection. Easley and O’Hara (1987), focusing on the learning process of market makers in dealer markets, found that the bid ask spread is positive related with the adverse selection risk.

Theoretical literature analyzing several market microstructures and touching on the problem of information asymmetry illustrates the advantages of private information. Kyle (1985) and Holden and Subrahmanyam (1992) both supposed that informed traders know the realized value of an asset. Admati and Pfleiderer (1988) then assumed that informed traders can obtain relevant information regarding asset value but do not completely know the asset value. Moreover, Foster and Viswanathan (1990) presumed public information as private
information with an additional random variable, disturbance. Since the uninformed trader is not able to differentiate which is private information and which is disturbance, this can show the advantages of private information and which also does not need to strongly assume that the informed trader completely knows everything about the asset value. It can be observed in this paper that public information is in fact private information with the addition of a disturbance.

Although all markets are closed at a predetermined time in the real world, traders still do not know if the market will be closed before the order execution when submitting an order.

The structure of this model is similar to Foucault’s (1999) wherein the order placement strategy is endogenous; hence there is no time reference from which we can start solving recursively for the equilibrium.

See Figure 2.

See Figure 2.

Foucault (1999) and Handa, Schwartz, and Tiwari (2003) also found that the probability of order execution affects the equilibrium price and bid-ask spread.

The U-shaped pattern of the average volume of shares traded has been documented in a number of studies, for example, by Jain and Joh (1986).

For examples, see Wood, McInish, and Ord (1985).

McInish and Wood (1992), Brock and Kleidon (1992), Lee, Mucklow, and Ready (1993), and Chan, Chung, and Johnson (1995) all showed that the spread of NYSE stocks is widest at the open, then drops sharply during the first hour of trading, and finally increases slightly before market close.

Appendixes

Appendix A

Proof of Proposition 1.
Given our framework, consider the optimal order placement decision of uninformed buyers. The price that they are indifferent between a buy market order or a buy limit order will satisfy

\[(V_u + \epsilon) + A^* = \rho \frac{1}{3} \{(1 - \pi_2)[3 - (p_1 + p_2 + p_3) + p_3 k_2] + (V_u + \epsilon) + B^*\}\]

The uninformed trader faces exactly the same type of problem, so we can write

\[B^* - (V_u + \epsilon) = \rho \frac{1}{3} \{(1 - \pi_1)[p_1(1 - k_1) + p_2 + p_3] + p_1 k_1\} (A^* - (V_u + \epsilon))\]
\[
x = \rho \frac{1}{3} \left[ (1 - \pi_2) \left[ 3 - (p_1 + p_2 + p_3) + p_3 k_2 \right] \right]
\]
\[
y = \rho \frac{1}{3} \left[ (1 - \pi_1) \left[ p_1 (1 - k_1) + p_2 + p_3 \right] + p_3 k_1 \right]
\]

(A3)  
(A4)

Equations (A1) and (A2) become
\[
(V_u + \varepsilon) - A^* = x \left\{ (V_u + \varepsilon) + B^* \right\}
\]
\[
B^* - (V_u - \varepsilon) = y \left\{ A^* - (V_u - \varepsilon) \right\}
\]

(A5)  
(A6)

Solving Eqs. (A5) and (A6), we obtain
\[
A^* = V_u + \frac{(1 + xy)\varepsilon - 2\varepsilon}{(1 - xy)}
\]
\[
B^* = V_u + \frac{2\varepsilon - (1 + xy)\varepsilon}{(1 - xy)}
\]
\[
A^* - B^* = 2\varepsilon \frac{(1 - x)(1 - y)}{(1 - xy)}
\]

(A7)  
(A8)  
(A9)

and
\[
x = \rho \frac{1}{3} \left[ (1 - \pi_2) \left[ 3 - (p_1 + p_2 + p_3) + p_3 k_2 \right] \right]
\]

(A10)

Since \( x \) is the average execution probability of a limit buy order, then
\[ 0 \leq x \leq 1. \]
\[
y = \rho \frac{1}{3} \left[ (1 - \pi_1) \left[ p_1 (1 - k_1) (p_2 + p_3) \right] + p_3 k_2 \right]
\]

(A11)

Since \( y \) is the average execution probability of a limit sell order, then
\[ 0 \leq y \leq 1, \text{ and } xy \leq 1. \]

By adding Eqs. (A10) and (A11), we obtain
\[
x + y \leq \frac{1}{3} \left[ 3 - (p_1 + p_2 + p_3) + p_3 k_2 \right] + \frac{1}{3} \left[ p_1 (1 - k_1) + p_2 + p_3 \right] + p_3 k_2 = 1
\]

(A12)
Appendix B

Proof of Proposition 2.

Using Eqs. (A7) and (A8), we obtain:

$$\frac{\partial A^*}{\partial \varepsilon} = \frac{(1 + xy - 2x)}{(1 - xy)} \geq 0$$  \hspace{1cm} (A13)

$$\frac{\partial A^*}{\partial x} = \frac{(1-xy)(y\varepsilon - 2\varepsilon) + [(1+xy)e - 2xe]\{e - y\}}{(1-xy)^2} \leq 0$$  \hspace{1cm} (A14)

$$\frac{\partial A^*}{\partial y} = \frac{(1-xy)(x\varepsilon) + [(1+xy)e - 2xe]\{e - x\}}{(1-xy)^2} \geq 0$$  \hspace{1cm} (A15)

$$\frac{\partial B^*}{\partial \varepsilon} = \frac{2y - (1+xy)}{(1 - xy)} \leq 0$$  \hspace{1cm} (A16)

$$\frac{\partial B^*}{\partial x} = \frac{(1-xy)(-y\varepsilon) + [2y\varepsilon - (1+xy)e]\{e - y\}}{(1-xy)^2} \leq 0$$  \hspace{1cm} (A17)

$$\frac{\partial B^*}{\partial y} = \frac{(1-xy)(2e - xe) + [2y\varepsilon - (1+xy)e]\{e - x\}}{(1-xy)^2} \geq 0$$  \hspace{1cm} (A18)

Appendix C

Proof of Proposition 3.

By the condition: $2y - 1 - xy \geq 0$ and equations from (A13) to (A18), we obtain

$$\frac{\partial A^*}{\partial \varepsilon} \geq 0 \quad \frac{\partial A^*}{\partial x} \leq 0 \quad \frac{\partial A^*}{\partial y} \geq 0 \quad \frac{\partial B^*}{\partial \varepsilon} \geq 0 \quad \frac{\partial B^*}{\partial x} \leq 0 \quad \frac{\partial B^*}{\partial y} \geq 0$$

Appendix D

Proof of Proposition 4.

By the condition: $1 + xy - 2x \leq 0$ and equations from (A13) to (A18), we obtain
\[ \frac{\partial A^*}{\partial \varepsilon} \leq 0, \quad \frac{\partial A^*}{\partial x} \leq 0, \quad \frac{\partial A^*}{\partial y} \geq 0 \quad \text{及} \quad \frac{\partial B^*}{\partial \varepsilon} \geq 0, \quad \frac{\partial B^*}{\partial x} \leq 0, \quad \frac{\partial B^*}{\partial y} \geq 0. \]

**Appendix E**

**Proof of Proposition 5.**

\[
\frac{\partial s}{\partial \varepsilon} = 2\left(\frac{(1-x)(1-y)}{(1-xy)}\right) \geq 0 \tag{A19}
\]

\[
\frac{\partial s}{\partial x} = \frac{(1-xy)(-2\varepsilon)(1-y) + 2\varepsilon(1-x)(1-y)y}{(1-xy)^3} \leq 0 \tag{A20}
\]

\[
\frac{\partial s}{\partial y} = \frac{(1-xy)(-2\varepsilon)(1-x) + 2\varepsilon(1-x)(1-y)x}{(1-xy)^2} \leq 0 \tag{A21}
\]

**Figures**

\[ V_1 < V_u \quad \text{ Buyers entry (} p_3 \text{) } \quad \text{Informed (} \hat{k}_1 \text{) } \quad \text{Market order (1)} \]

\[ V_1 = V_u \quad \text{ Buyers entry (} p_2 \text{) } \quad \text{Market order (1-} \pi_1 \text{) } \quad \text{Market order (1-} \pi_2 \text{) } \quad \text{Limit order (} \pi_1 \text{) } \quad \text{Limit order (} \pi_2 \text{) } \]

\[ V_1 > V_u \quad \text{ Sellers entry (} 1- p_1 \text{) } \quad \text{Informed (} \hat{k}_1 \text{) } \quad \text{Market order (1-} \pi_1 \text{) } \quad \text{Market order (1-} \pi_2 \text{) } \quad \text{Limit order (} \pi_1 \text{) } \quad \text{Limit order (} \pi_2 \text{) } \]

\[ V_1 < V_u \quad \text{ Buyers entry (} p_3 \text{) } \quad \text{Market order (1)} \quad \text{Market order (1-} \pi_1 \text{) } \quad \text{Limit order (} \pi_1 \text{) } \quad \text{Informed (} \hat{k}_2 \text{) } \quad \text{uninformed (1-} \hat{k}_2 \text{) } \quad \text{Limit order (} \pi_2 \text{) } \]

**Figure 1. The order path faced by uninformed traders.**
Note: Horizontal axis x is the average probability of an execution of a limit buy order, and vertical axis y is the average probability of execution of a limit sell order.

Figure 2. The space of x and y.

Reference


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委託單執行的動態及限價單成本的日內型態分析

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本文建構了一個委託單驅動市場的動態模型來分析交易者下限價單的決策，假定交易者對資產的評價不同，而在考慮了非資訊交易者的學習過程後，我們可以分析委託單成交的動態。本文的模型推論與既存的討論委託單序列及執行之實證文獻密切相關，並補足了文獻對限價委託單成本討論的缺乏：文中指出非資訊交易者之限價單交易成本的日內型態為U型。

關鍵字：委託單驅動市場、成交機率、委託單序列、日內型態、執行成本